$y - y_0$.

Date.	·0.	·1.	·2.	·3.	·4.	·5.	·6.	·7.	·8.	·9.
1907	·134	·053	−·035	−·113	−·168	−·198	−·138	−·050	·048	·143
08	·205	·216	·156	·051	−·064	−·168	−·216	−·198	−·149	−·069
09	·041	·145	·220	·242	·298	·079	−·049	−·147	−·204	−·233
10	−·222	−·101	·030	·156	·222	·244	·191	·055	−·077	−·190
11	−·259	−·249	−·167	−·048	·104	·220	·278	·263	·175	·050
12	−·090	−·157	−·205	−·191	−·131	−·023	·084	·160	·213	·223
13	·169	·096	·004	−·097	−·161	−·170	−·139	−·081	−·005	·074

These co-ordinates give an indication of a term with a period of about 15 months, as well as the 14-monthly term. The apparent amplitude of the 14-monthly motion shows a variation in consequence of this, but the series of observations is not long enough to make it possible to determine either the period or the amplitude of this term satisfactorily. The existence of a term whose period is 15·98 months has been indicated in the azimuth errors of the Cape Transit Circle by Professor Turner.* The effects of any variation in the amplitudes of the annual motion will also be included in the residuals of these tables.

The Kinetic Energy of a Star Cluster. By A. S. Eddington, M.A., M.Sc., F.R.S., Plumian Professor.

1. *In any Star Cluster in a steady state the Internal Kinetic Energy is one-half the exhaustion of Potential Energy.*—Although I cannot find that this result has ever been explicitly stated, it follows almost immediately from formulæ found in the theory of gases in connection with the virial.† It will probably be more convenient to astronomers to give in full the proof for stars, instead of quoting these formulæ. It will be seen that the result does not in any way depend on the special conditions of a gas, but is a perfectly general consequence of the law of gravitation.

Taking the centre of gravity of the cluster as origin, let

x, y, z be the co-ordinates of a star of mass m,

X, Y, Z the gravitational force on m, due to the attractions of the other stars of the cluster,

T the total kinetic energy of the motions of the stars relative to the centre of gravity,

Ω the exhaustion of potential energy of the cluster,

C the moment of inertia of the cluster about its centre of gravity.

* Turner, *Tables for facilitating Harmonic Analysis*, p. 41.

† *Cf.* Jeans, *Dynamical Theory of Gases*, first edition, p. 145, where the virial is found to $-\frac{1}{2}\Sigma\Sigma r\phi(r)$. Put $\phi(r) = -1/r^2$, and the result follows immediately. I have since learnt that the result has been given by H. Poincaré (*Hypothèses cosmogoniques*, p. 94) in a review of Ligondés' hypothesis of a swarm of meteorites. I am indebted to Mr. H. Jeffreys for this reference.

We have
$$m\frac{d^2x}{dt^2}=\mathrm{X}.$$

From the identity
$$\tfrac{1}{2}\frac{d^2}{dt^2}(x^2)=\left(\frac{dx}{dt}\right)^2+x\frac{d^2x}{dt^2}$$
we have
$$\tfrac{1}{4}m\frac{d^2}{dt^2}(x^2)=\tfrac{1}{2}m\left(\frac{dx}{dt}\right)^2+\tfrac{1}{2}x\mathrm{X}.$$

Add this equation and the two corresponding equations for y and z, and sum for all the stars of the cluster; the result is
$$\tfrac{1}{4}\Sigma m\frac{d^2}{dt^2}(r^2)=\tfrac{1}{2}\Sigma mv^2+\tfrac{1}{2}\Sigma(x\mathrm{X}+y\mathrm{Y}+z\mathrm{Z}) \quad . \quad . \quad (1)$$
where $r^2=x^2+y^2+z^2$, and v is the velocity of the star.

The left-hand side
$$=\tfrac{1}{4}\frac{d^2}{dt^2}\Sigma mr^2$$
$$=\tfrac{1}{4}\frac{d^2\mathrm{C}}{dt^2}$$

This evidently vanishes in a steady state, because the moment of inertia cannot be altering. Equation (1) therefore reduces to
$$\mathrm{T}=-\tfrac{1}{2}\Sigma(x\mathrm{X}+y\mathrm{Y}+z\mathrm{Z}) \quad . \quad . \quad . \quad (2)$$

To evaluate (2), we remark that X is the sum of the attractions on a particular star of all the other stars, so that the whole expression may be split up into separate terms corresponding to every possible pair of stars. Consider a pair of stars m_1 and m_2. The contribution due to the attraction of m_2 on m_1 is
$$-\tfrac{1}{2}x_1\frac{m_1m_2(x_2-x_1)}{r_{12}^{\,3}}-\tfrac{1}{2}y_1\frac{m_1m_2(y_2-y_1)}{r_{12}^{\,3}}-\tfrac{1}{2}z_1\frac{m_1m_2(z_2-z_1)}{r_{12}^{\,3}}$$
where
$$r_{12}^{\,2}=(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2.$$

The portion due to the attraction of m_1 on m_2 is
$$-\tfrac{1}{2}x_2\frac{m_1m_2(x_1-x_2)}{r_{12}^{\,3}}-\tfrac{1}{2}y_2\frac{m_1m_2(y_1-y_2)}{r_{12}^{\,3}}-\tfrac{1}{2}z_2\frac{m_1m_2(z_1-z_2)}{r_{12}^{\,3}}.$$

Adding these together, they give
$$\tfrac{1}{2}\frac{m_1m_2}{r_{12}}.$$

Hence
$$-\tfrac{1}{2}\Sigma(x\mathrm{X}+y\mathrm{Y}+z\mathrm{Z})=\tfrac{1}{2}\Sigma\frac{m_1m_2}{r_{12}},$$
the latter sum being taken over all combinations of stars in pairs.
$$=-\tfrac{1}{2}\text{ (potential energy of cluster).}$$

Hence (2) gives the result enunciated, namely,

$$T = \tfrac{1}{2}\Omega \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The whole energy of the cluster is $T - \Omega$, and the result may be expressed in the form—

Potential energy is twice the whole energy.

Kinetic energy is minus the whole energy.

2. It is of some interest to notice the general formula when the state is not steady. Equation (1) then gives

$$\tfrac{1}{4}\frac{d^2C}{dt^2} = T - \tfrac{1}{2}\Omega \quad . \quad . \quad . \quad . \quad (4)$$

so that the result holds for a roughly steady state, if d^2C/dt^2 is negligible.

Moreover, if $T > \frac{1}{2}\Omega$, d^2C/dt^2 is positive, so that the cluster expands (or its rate of contraction diminishes so that there is ultimately an expansion). Usually expansion diminishes Ω, and consequently diminishes $T - \frac{1}{2}\Omega$, the whole energy $T - \Omega$ remaining constant. Similarly, if $T < \frac{1}{2}\Omega$, there is a tendency for $T - \frac{1}{2}\Omega$ to increase. We see, therefore, that the value of $T - \frac{1}{2}\Omega$ is likely to be accelerated towards zero; but, since there is no absolute relation between C and Ω, we cannot draw any definite conclusion without further data as to the constitution of the cluster. Perhaps the cluster will attain the steady state by a series of rapidly damped oscillations of $T - \frac{1}{2}\Omega$.

3. *Rate of Dissolution of a Moving Cluster.*—This example is appended to illustrate the use that can be made of the foregoing result. It has been shown by Jeans * that the parallelism of the motions of stars in a cluster is destroyed with exceeding slowness by perturbations due to the passage of other stars through the cluster. But although the deflections of motion are very small, they will lead to a comparatively rapid scattering of the stars in space unless counteracted by some other force such as the mutual attraction of the cluster. For example, from the present dimensions of the Taurus cluster I have shown that it could not have existed more than 57 million years unless the scattering had been counteracted.† We are now in a position to take this mutual attraction into account and estimate the actual rate of scattering.

The chance perturbations by external stars produce a probable deflection of the motion of any cluster star proportional to the square-root of the time, and therefore give it a probable kinetic energy (relative to the centroid of the cluster) proportional to the time elapsed. This internal energy is acquired at the expense of the translational energy of the cluster as a whole or of the stars encountered. Let M be the mass of the whole cluster, a the velocity relative to the cluster acquired by a star in unit time (mean-square value); then the cluster will be gaining internal

* *Monthly Notices*, vol. lxxiv. p. 109.

† *Stellar Movements and the Structure of the Universe*, p. 254.

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energy at the rate $\frac{1}{2}M\alpha^2$. By § 1 the rate of increase of potential energy of the cluster will then be $M\alpha^2$, provided an approximately steady state is maintained. (Since the cluster stars are closer and their relative motions much slower than the external stars, their mutual encounters will be more potent as a means of transferring energy; that is to say, the energy will be redistributed within the cluster more rapidly than it is acquired from outside, and a steady state should result.) Now the potential energy of the cluster is $-M^2/2c$, where c is proportional to the linear dimensions of the cluster and is of the same order of magnitude as the average radius. For Plummer's law of density, $(1+r^2)^{-\frac{5}{2}}$, c is about 0·82 times the median radius of the cluster. Thus we have

$$\frac{d}{dt}\left(-\frac{M^2}{2c}\right)=M\alpha^2 \quad . \quad . \quad . \quad . \quad (5)$$

or, integrating for an interval t,

$$\frac{1}{c_0}-\frac{1}{c}=\frac{2\alpha^2 t}{M}.$$

The time taken for the cluster to increase from half its present linear dimensions to its present size is thus

$$\tau=\frac{M}{2c\alpha^2} \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Taking Jeans's figures for the Taurus cluster, he finds an average deflection of 1′ in a million years, which, for a velocity of the cluster of 40 km./sec., is equivalent to a transverse velocity of 0·012 parsecs per million years. If we take for the unit of mass the sun's mass, and for the unit of time a million years, the unit of length must be 0·165 parsecs if the gravitational constant is eliminated. Hence $\alpha=0·073$. We may take c as roughly 3 parsecs, or 18 units. Then (6) gives

$$\tau=M\times 5 \text{ million years.}$$

Forty stars are known to belong the cluster, and there are doubtless many others. Thus τ must be several hundred million years at least, and the present concentration of the cluster does *not* indicate an unduly short period of existence.